

Algebraic Number Theory

(PARI-GP version 2.19.0)

Binary Quadratic Forms

create $ax^2 + bxy + cy^2$ **Qfb**(a, b, c) or **Qfb**($[a, b, c]$)
reduce x ($s = \sqrt{D}$, $l = \lfloor s \rfloor$) **qfbred**($x, \{flag\}, \{D\}, \{l\}, \{s\}$)
return $[y, g]$, $g \in \text{SL}_2(\mathbf{Z})$, $y = g \cdot x$ reduced **qfbreds12**(x)
composition of forms $x*y$ or **qfbnucomp**(x, y, l)
 n -th power of form x^n or **qfbnupow**(x, n)
composition **qfbcomp**(x, y)
... without reduction **qfbcomppraw**(x, y)
 n -th power **qfbpow**(x, n)
... without reduction **qfbpowraw**(x, n)
prime form of disc. x above prime p **qfbprimeform**(x, p)
class number of disc. x **qfbclassno**(x)
Hurwitz class number of disc. x **qfbhclassno**(x)
solve $Q(x, y) = n$ in integers **qfbsolve**(Q, n)
solve $x^2 + Dy^2 = p$, p prime **qfbcornacchia**(D, p)
... $x^2 + Dy^2 = 4p$, p prime **qfbcornacchia**($D, 4 * p$)

Quadratic Fields

quadratic number $\omega = \sqrt{x}$ or $(1 + \sqrt{x})/2$ **quadgen**(x)
minimal polynomial of ω **quadpoly**(x)
discriminant of $\mathbf{Q}(\sqrt{x})$ **quaddisc**(x)
regulator of real quadratic field **quadregulator**(x)
fundamental unit in O_D , $D > 0$ **quadunit**($D, \{w\}$)
norm of fundamental unit in O_D **quadunitnorm**(D)
index of $O_{Df^2}^\times$ in $O_D^\times$ **quadunitindex**(D, f)
class group of $\mathbf{Q}(\sqrt{D})$ **quadclassunit**($D, \{flag\}, \{t\}$)
Hilbert class field of $\mathbf{Q}(\sqrt{D})$ **quadhilbert**($D, \{flag\}$)
... using specific class invariant ($D < 0$) **polclass**($D, \{inv\}$)
test if T is **polclass**(D); if so return D **polisclass**(T)
ray class field modulo f of $\mathbf{Q}(\sqrt{D})$ **quadray**($D, f, \{flag\}$)

General Number Fields: Initializations

The number field $K = \mathbf{Q}[X]/(f)$ is given by irreducible $f \in \mathbf{Q}[X]$.
We denote $\theta = \bar{X}$ the canonical root of f in K . A nf structure contains a maximal order and allows operations on elements and ideals. A bnf adds class group and units. A bnr is attached to ray class groups and class field theory. A rmf is attached to relative extensions L/K .

init number field structure nf **nfinit**($f, \{flag\}$)
known integer basis B **nfinit**($[f, B]$)
order maximal at $vp = [p_1, \dots, p_k]$ **nfinit**($[f, vp]$)
order maximal at all $p \leq P$ **nfinit**($[f, P]$)
certify maximal order **nfcertify**(nf)

nf members:

a monic $F \in \mathbf{Z}[X]$ defining K **nf.pol**
number of real/complex places **nf.r1/r2/sign**
discriminant of nf **nf.disc**
primes ramified in nf **nf.p**
 T_2 matrix **nf.t2**
complex roots of F **nf.roots**
integral basis of $\mathbf{Z}_K$ as powers of θ **nf.zk**
different/codifferent **nf.diff**, **nf.codiff**
index $[\mathbf{Z}_K : \mathbf{Z}[X]/(F)]$ **nf.index**
recompute nf using current precision **nfnewprec**(nf)
init relative rmf $L = K[Y]/(g)$ **rnfininit**(nf, g)
init bnf structure **bnfinit**(f, l)

bnf members: same as nf , plus
underlying nf **bnf.nf**
class group, regulator **bnf.clgp**, **bnf.reg**
fundamental/torsion units **bnf.fu**, **bnf.tu**
add S -class group and units, yield $bnfS$ **bnfsunit**(bnf, S)
init class field structure bnr **bnrinit**($bnf, m, \{flag\}$)
bnr members: same as bnf , plus
underlying bnf **bnr.bnf**
big ideal structure **bnr.bid**
modulus m **bnr.mod**
structure of $(\mathbf{Z}_K/m)^*$ **bnr.zkst**

Fields, subfields, embeddings

Defining polynomials, embeddings
(some) number fields with Galois group G **nflist**(G)
... and $|\text{disc}(K)| = N$ and s complex places **nflist**($G, N, \{s\}$)
... and $a \leq |\text{disc}(K)| \leq b$ **nflist**($G, [a, b], \{s\}$)
smallest poly defining $f = 0$ (slow) **polredabs**($f, \{flag\}$)
small poly defining $f = 0$ (fast) **polredbest**($f, \{flag\}$)
monic integral $g = Cf(x/L)$ **polmonic**($f, \{\&L\}$)
random Tschirnhausen transform of f **poltschirnhaus**(f)
 $\mathbf{Q}[t]/(f) \subset \mathbf{Q}[t]/(g)$? Isomorphic? **nfisincl**(f, g), **nfisisom**
reverse polmod $a = A(t) \bmod T(t)$ **modreverse**(a)
compositum of $\mathbf{Q}[t]/(f)$, $\mathbf{Q}[t]/(g)$ **polcompositum**($f, g, \{flag\}$)
compositum of $K[t]/(f)$, $K[t]/(g)$ **nfcompositum**($nf, f, g, \{flag\}$)
splitting field of K (degree divides d) **nfsplitting**($nf, \{d\}$)
signs of real embeddings of x **nfeltsign**($nf, x, \{pl\}$)
complex embeddings of x **nfeltembed**($nf, x, \{pl\}$)
 $T \in K[t]$, # of real roots of $\sigma(T) \in R[t]$ **nfpolsturm**($nf, T, \{pl\}$)
absolute Weil height **nfweilheight**(nf, v)

Subfields, polynomial factorization

subfields (of degree d) of nf **nfsubfields**($nf, \{d\}$)
maximal subfields of nf **nfsubfieldsmax**(nf)
maximal CM subfield of nf **nfsubfieldscm**(nf)
 $K_d \subset \mathbf{Q}(\zeta_n)$, using Gaussian periods **polsubcyclo**($n, d, \{v\}$)
... using class field theory **polsubcyclofast**(n, d)
roots of unity in nf **nfroots1**(nf)
roots of g belonging to nf **nfroots**(nf, g)
factor g in nf **nfactor**(nf, g)

Linear and algebraic relations

poly of degree $\leq k$ with root $x \in \mathbf{C}$ or $\mathbf{Q}_p$ **algdep**(x, k)
alg. dep. with pol. coeffs for series s **seralgdep**(s, x, y)
diff. dep. with pol. coeffs for series s **serdiffdep**(s, x, y)
small linear rel. on coords of vector x **lindep**(x)

Basic Number Field Arithmetic (nf)

Number field elements are **t_INT**, **t_FRAC**, **t_POL**, **t_POLMOD**, or **t_COL** (on integral basis $nf.zk$).

Basic operations

$x + y$ **nfeltadd**(nf, x, y)
 $x \times y$ **nfeltmul**(nf, x, y)
 x^n , $n \in \mathbf{Z}$ **nfeltpow**(nf, x, n)
 x/y **nfeltdiv**(nf, x, y)
 $q = x \setminus y := \text{round}(x/y)$ **nfeltdivu**(nf, x, y)
 $r = x \% y := x - (x \setminus y)y$ **nfeltmod**(nf, x, y)
... $[q, r]$ as above **nfeltdivrem**(nf, x, y)
reduce x modulo ideal A **nfeltreduce**(nf, x, A)
absolute trace $\text{Tr}_{K/\mathbf{Q}}(x)$ **nfelttrace**(nf, x)
absolute norm $N_{K/\mathbf{Q}}(x)$ **nfeltnorm**(nf, x)

is x a square? **nfeltissquare**($nf, x, \{\&y\}$)
... an n -th power? **nfeltispower**($nf, x, n, \{\&y\}$)

Multiplicative structure of K^* ; $K^*/(K^*)^n$

valuation $v_{\mathfrak{p}}(x)$ **nfeltval**($nf, x, \mathfrak{p}$)
... write $x = \pi^{v_{\mathfrak{p}}(x)}y$ **nfeltval**($nf, x, \mathfrak{p}, \{\&y\}$)
quadratic Hilbert symbol (at $\mathfrak{p}$) **nfhilbert**($nf, a, b, \{\mathfrak{p}\}$)
 b such that $xb^n = v$ is small **idealredmodpower**(nf, x, n)

Maximal order and discriminant

integral basis of field $\mathbf{Q}[x]/(f)$ **nfbasis**(f)
field discriminant of $\mathbf{Q}[x]/(f)$ **nfdisc**(f)
... and factorization **nfdiscfactors**(f)
express x on integer basis **nfalgtobasis**(nf, x)
express element x as a polmod **nfbasistoalg**(nf, x)

Hecke Grossencharacters

Let K be a number field and m a modulus. A **gchar** structure describes the group of Hecke Grossencharacters of K of modulus m and allows computations with these characters. A character χ is described by its components modulo $gc.cyc$.

init **gchar** structure gc for modulus m **gcharinit**($bnf, m, \{cm\}$)

gc members:

underlying bnf **gc.bnf**
modulus **gc.mod**
elementary divisors (including 0s) **gc.cyc**
recompute gc using current precision **gcharnewprec**(gc)
evaluate Hecke character chi at ideal id **gchareval**(gc, chi, id)
exponent column of id in $\mathbf{R}^n$ **gcharideallog**(gc, id)
log representation of ideal id **gcharlog**(gc, id)
... of character χ **gcharduallog**(gc, chi)
exponent vector of χ in $\mathbf{R}^n$ **gcharparameters**(gc, chi)
conductor of χ **gcharconductor**(gc, chi)
L-function of χ **lfuncreate**($[gc, chi]$)
local component χ_v of χ **gcharlocal**(gc, chi, v)
 χ s.t. $\chi_v \approx Lchiv[i]$ for $v = Lv[i]$ **gcharidentify**($gc, Lv, Lchiv$)
basis of group of algebraic characters **gcharalgebraic**(gc)
algebraic character of given infinity type **gcharalgebraic**($gc, type$)
is χ algebraic? **gcharisalgebraic**(gc, chi)

Dedekind Zeta Function ζ_K , Hecke L series

$R = [c, w, h]$ in initialization means we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w$, $|\Im(s)| < h$; $R = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $R = [1/2, 0, h]$ (critical line up to height h).

ζ_K as Dirichlet series, $N(I) \leq b$ **dirzetak**(nf, b)
init $\zeta_K^{(k)}(s)$ for $k \leq n$ **L = lfunitinit**($bnf, R, \{n = 0\}$)
compute $\zeta_K(s)$ (n -th derivative) **lfun**($L, s, \{n = 0\}$)
compute $\Lambda_K(s)$ (n -th derivative) **lfunlambda**($L, s, \{n = 0\}$)

init $L_K^{(k)}(s, \chi)$ for $k \leq n$ **L = lfunitinit**($[bnr, chi], R, \{n = 0\}$)
compute $L_K(s, \chi)$ (n -th derivative) **lfun**($L, s, \{n\}$)
Artin root number of K **bnrrootnumber**($bnr, chi, \{flag\}$)
 $L(1, \chi)$, for all χ trivial on H **bnrL1**($bnr, \{H\}, \{flag\}$)

Class Groups & Units (bnf, bnr)

Class field theory data $a_1, \{a_2\}$ is usually bnr (ray class field), bnr, H (congruence subgroup) or bnr, χ (character on **bnr.clgp**). Any of these define a unique abelian extension of K .
units / S -units **bnfunits**($bnf, \{S\}$)
remove GRH assumption from bnf **bnfcertify**(bnf)

expo. of ideal x on class gp	<code>bnfisprincipal(<i>bnf</i>, x, {<i>flag</i>})</code>
...on ray class gp	<code>bnrisprincipal(<i>bnr</i>, x, {<i>flag</i>})</code>
expo. of x on fund. units	<code>bnfisunit(<i>bnf</i>, x)</code>
...on S -units, U is <code>bnfunits(<i>bnf</i>, S)</code>	<code>bnfisunit(<i>bnfs</i>, x, U)</code>
signs of real embeddings of <i>bnf</i> .fu	<code>bnfsignunit(<i>bnf</i>)</code>
narrow class group	<code>bnfnarrow(<i>bnf</i>)</code>

Class Field Theory

ray class number for modulus m	<code>bnrclassno(<i>bnf</i>, m)</code>
discriminant of class field	<code>bnrdisc(a_1, {a_2})</code>
ray class numbers, l list of moduli	<code>bnrclassnolist(<i>bnf</i>, l)</code>
discriminants of class fields	<code>bnrdisclist(<i>bnf</i>, l, {<i>arch</i>}, {<i>flag</i>})</code>
decode output from <code>bnrdisclist</code>	<code>bnfdecodemodule(<i>nf</i>, f, a)</code>
is modulus the conductor?	<code>bnrisconductor(a_1, {a_2})</code>
is class field (bnr , H) Galois over K^G	<code>bnrisgalois(<i>bnr</i>, G, H)</code>
action of automorphism on <code>bnr.gen</code>	<code>bnrgaloismatrix(<i>bnr</i>, aut)</code>
apply <code>bnrgaloismatrix</code> M to H	<code>bnrgaloisapply(<i>bnr</i>, M, H)</code>
characters on <code>bnr.clgp</code> s.t. $\chi(g_i) = e(v_i)$	<code>bnrchar(<i>bnr</i>, g, {v})</code>
conductor of character χ	<code>bnrconductor(<i>bnr</i>, chi)</code>
conductor of extension	<code>bnrconductor(a_1, {a_2}, {<i>flag</i>})</code>
conductor of extension $K[Y]/(g)$	<code>rnfconductor(<i>bnf</i>, g)</code>
canonical projection $Cl_F \rightarrow Cl_f$, $f \mid F$	<code>bnrmap</code>
Artin group of extension $K[Y]/(g)$	<code>rnfnormgroup(<i>bnr</i>, g)</code>
subgroups of bnr , index $\leq b$	<code>subgrouplist(<i>bnr</i>, b, {<i>flag</i>})</code>
compositum as <code>[bnr,H]</code>	<code>bnrcompositum([<i>bnr</i>1, H1], [<i>bnr</i>2, H2])</code>
class field defined by $H \subset Cl_f$	<code>bnrclassfield(<i>bnr</i>, H)</code>
...low level equivalent, prime degree	<code>rnfkummer(<i>bnr</i>, H)</code>
same, using Stark units (real field)	<code>bnrstark(<i>bnr</i>, {<i>sub</i>}, {<i>flag</i>})</code>
Stark unit	<code>bnrstarkunit(<i>bnr</i>, {<i>sub</i>})</code>
is a an n -th power in K_v ?	<code>nfislocalpower(<i>nf</i>, v, a, n)</code>
cyclic L/K satisf. local conditions	<code>nfgrunwaldwang(<i>nf</i>, P, D, p)</code>

Cyclotomic and Abelian fields theory

An Abelian field F given by a subgroup $H \subset (Z/fZ)^*$ is described by an argument F , e.g. f (for $H = 1$, i.e. $Q(\zeta_f)$) or $[G, H]$, where G is `idealstar( $f$ , 1)`, or a minimal polynomial.

minus class number $h^-(F)$	<code>subcyclohminus(F)</code>
... p -part	<code>subcyclohminus(F, p)</code>
minus part of Iwasawa polynomials	<code>subcycloiwawasa(F, p)</code>
p -Sylow of $Cl(F)$	<code>subcyclopclgp(F, p)</code>
is p coprime to $h(Q(\zeta_p))$?	<code>primeisregular(p)</code>

Logarithmic class group

logarithmic ℓ -class group	<code>bnflog(<i>bnf</i>, ℓ)</code>
$[\tilde{e}(F_v/Q_p), \tilde{f}(F_v/Q_p)]$	<code>bnflogef(<i>bnf</i>, pr)</code>
$\exp \deg_F(A)$	<code>bnflogdegree(<i>bnf</i>, A, ℓ)</code>
is ℓ -extension L/K locally cyclotomic	<code>rnfislocalcyclo(<i>rnf</i>)</code>

Ideals: elements, primes, or matrix of generators in HNF

is id an ideal in nf ?	<code>nfisideal(<i>nf</i>, id)</code>
is x principal in bnf ?	<code>bnfisprincipal(<i>bnf</i>, x)</code>
give $[a, b]$, s.t. $a\mathbf{Z}_K + b\mathbf{Z}_K = x$	<code>idealtwoelt(<i>nf</i>, x, {a})</code>
put ideal a ($a\mathbf{Z}_K + b\mathbf{Z}_K$) in HNF	<code>idealhnf(<i>nf</i>, a, {b})</code>
PNF for ideal generated by $a_1, \dots, a_n$	<code>idealfromgens(<i>nf</i>, a)</code>
norm of ideal x	<code>idealnrm(<i>nf</i>, x)</code>
minimum of ideal x (direction v)	<code>idealmin(<i>nf</i>, x, v)</code>
LLL-reduce the ideal x (direction v)	<code>idealred(<i>nf</i>, x, {v})</code>

Algebraic Number Theory

(PARI-GP version 2.19.0)

Ideal Operations

add ideals x and y	<code>idealadd(<i>nf</i>, x, y)</code>
multiply ideals x and y	<code>idealmul(<i>nf</i>, x, y, {<i>flag</i>})</code>
intersection of ideal x with Q	<code>idealdown(<i>nf</i>, x)</code>
intersection of ideals x and y	<code>idealintersect(<i>nf</i>, x, y, {<i>flag</i>})</code>
n -th power of ideal x	<code>idealpow(<i>nf</i>, x, n, {<i>flag</i>})</code>
inverse of ideal x	<code>idealinv(<i>nf</i>, x)</code>
divide ideal x by y	<code>idealdiv(<i>nf</i>, x, y, {<i>flag</i>})</code>
Find $(a, b) \in x \times y$, $a + b = 1$	<code>idealaddtoone(<i>nf</i>, x, {y})</code>
coprime integral A, B such that $x = A/B$	<code>idealnumden(<i>nf</i>, x)</code>

Primes and Multiplicative Structure

check whether x is a maximal ideal	<code>idealismaximal(<i>nf</i>, x)</code>
factor ideal x in $\mathbf{Z}_K$	<code>idealfactor(<i>nf</i>, x)</code>
expand ideal factorization in K	<code>idealfactorback(<i>nf</i>, f, {e})</code>
is ideal A an n -th power ?	<code>idealispower(<i>nf</i>, A, n)</code>
expand elt factorization in K	<code>nffactorback(<i>nf</i>, f, {e})</code>
decomposition of prime p in $\mathbf{Z}_K$	<code>idealprimedec(<i>nf</i>, p)</code>
valuation of x at prime ideal pr	<code>idealval(<i>nf</i>, x, pr)</code>
weak approximation theorem in nf	<code>idealchinese(<i>nf</i>, x, y)</code>
$a \in K$, s.t. $v_p(a) = v_p(x)$ if $v_p(x) \neq 0$	<code>idealappr(<i>nf</i>, x)</code>
$a \in K$ such that $(a \cdot x, y) = 1$	<code>idealcoprime(<i>nf</i>, x, y)</code>
give bid =structure of $(\mathbf{Z}_K/id)^*$	<code>idealstar(<i>nf</i>, id, {<i>flag</i>})</code>
structure of $(1 + \mathfrak{p})/(1 + \mathfrak{p}^k)$	<code>idealprincipalunits(<i>nf</i>, pr, k)</code>
discrete log of x in $(\mathbf{Z}_K/bid)^*$	<code>ideallog(<i>nf</i>, x, bid)</code>
<code>idealstar</code> of all ideals of norm $\leq b$	<code>ideallist(<i>nf</i>, b, {<i>flag</i>})</code>
add Archimedean places	<code>ideallistarch(<i>nf</i>, b, {<i>ar</i>}, {<i>flag</i>})</code>
init <code>modpr</code> structure	<code>nfmodprinit(<i>nf</i>, pr, {v})</code>
project t to $\mathbf{Z}_K/pr$	<code>nfmodpr(<i>nf</i>, t, <i>modpr</i>)</code>
lift from $\mathbf{Z}_K/pr$	<code>nfmodprlift(<i>nf</i>, t, <i>modpr</i>)</code>

Galois theory over Q

conjugates of a root θ of nf	<code>nfgaloisconj(<i>nf</i>, {<i>flag</i>})</code>
apply Galois automorphism s to x	<code>nfgaloisapply(<i>nf</i>, s, x)</code>
Galois group of field $\mathbf{Q}[x]/(f)$	<code>polgalois(f)</code>
resolvent field of $\mathbf{Q}[x]/(f)$	<code>nfresolvent(f)</code>
initializes a Galois group structure G	<code>galoisinit(<i>pol</i>, {<i>den</i>})</code>
...for the splitting field of pol	<code>galoissplittinginit(<i>pol</i>, {d})</code>
character table of G	<code>galoischartable(G)</code>
conjugacy classes of G	<code>galoisconjclasses(G)</code>
$\det(1 - \rho(g)T)$, χ character of ρ	<code>galoischarpoly(G, χ, {o})</code>
$\det(\rho(g))$, χ character of ρ	<code>galoischardet(G, χ, {o})</code>
action of p in <code>nfgaloisconj</code> form	<code>galoispermtpol(G, {p})</code>
identify as abstract group	<code>galoisidentify(G)</code>
export a group for GAP/MAGMA	<code>galoisexport(G, {<i>flag</i>})</code>
subgroups of the Galois group G	<code>galoissubgroups(G)</code>
order of a group or subgroup	<code>galoisorder(H)</code>
is subgroup H normal?	<code>galoisisnormal(G, H)</code>
subfields from subgroups	<code>galoissubfields(G, {<i>flag</i>}, {v})</code>
fixed field	<code>galoisfixedfield(G, <i>perm</i>, {<i>flag</i>}, {v})</code>
Frobenius at maximal ideal P	<code>idealfrobenius(<i>nf</i>, G, P)</code>
ramification groups at P	<code>idealramgroups(<i>nf</i>, G, P)</code>
is G abelian?	<code>galoisisabelian(G, {<i>flag</i>})</code>
abelian number fields/ $\mathbf{Q}$	<code>galoissubcyclo(N, H, {<i>flag</i>}, {v})</code>

The galpol package

query the package: polynomial	<code>galoisgetpol(a, b, {s})</code>
...: permutation group	<code>galoisgetgroup(a, b)</code>
...: group description	<code>galoisgetname(a, b)</code>

Relative Number Fields (rnf)

Extension L/K is defined by $T \in K[x]$.

absolute equation of L	<code>rnfequation(<i>nf</i>, T, {<i>flag</i>})</code>
is L/K abelian?	<code>rnfisabelian(<i>nf</i>, T)</code>
relative <code>nfalgtobasis</code>	<code>rnfalgtobasis(<i>rnf</i>, x)</code>
relative <code>nfbasistoalg</code>	<code>rnfbasistoalg(<i>rnf</i>, x)</code>
relative <code>idealhnf</code>	<code>rnfidealhnf(<i>rnf</i>, x)</code>
relative <code>idealmul</code>	<code>rnfidealmul(<i>rnf</i>, x, y)</code>
relative <code>idealtwoelt</code>	<code>rnfidealtwoelt(<i>rnf</i>, x)</code>

Lifts and Push-downs

absolute $\rightarrow$ relative representation for x	<code>rnfeltabstorel(<i>rnf</i>, x)</code>
relative $\rightarrow$ absolute representation for x	<code>rnfeltretloabs(<i>rnf</i>, x)</code>
lift x to the relative field	<code>rnfeltup(<i>rnf</i>, x)</code>
push x down to the base field	<code>rnfeltdown(<i>rnf</i>, x)</code>
idem for x ideal: (<code>rnfideal</code>) <code>reltoabs</code> , <code>abstorel</code> , <code>up</code> , <code>down</code>	

Norms and Trace

relative norm of element $x \in L$	<code>rnfeltnrm(<i>rnf</i>, x)</code>
relative trace of element $x \in L$	<code>rnfelttrace(<i>rnf</i>, x)</code>
absolute norm of ideal x	<code>rnfidealnrmabs(<i>rnf</i>, x)</code>
relative norm of ideal x	<code>rnfidealnrmrel(<i>rnf</i>, x)</code>
solutions of $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$	<code>bnfisintnrm(<i>bnf</i>, x)</code>
is $x \in \mathbf{Q}$ a norm from K ?	<code>bnfisnorm(<i>bnf</i>, x, {<i>flag</i>})</code>
initialize T for norm eq. solver	<code>rnfisnorminit(K, <i>pol</i>, {<i>flag</i>})</code>
is $a \in K$ a norm from L ?	<code>rnfisnorm(T, a, {<i>flag</i>})</code>
initialize t for Thue equation solver	<code>thueinit(f)</code>
solve Thue equation $f(x, y) = a$	<code>thue(t, a, {<i>sol</i>})</code>
characteristic poly. of $a \bmod T$	<code>rnfcharpoly(<i>nf</i>, T, a, {v})</code>

Factorization

factor ideal x in L	<code>rnfidealfactor(<i>rnf</i>, x)</code>
$[S, T]: T_{i,j} \mid S_i$; S primes of K above p	<code>rnfidealprimedec(<i>rnf</i>, p)</code>

Maximal order $\mathbf{Z}_L$ as a $\mathbf{Z}_K$ -module

relative <code>polredbest</code>	<code>rnfpolredbest(<i>nf</i>, T)</code>
relative <code>polredabs</code>	<code>rnfpolredabs(<i>nf</i>, T)</code>
relative Dedekind criterion, prime pr	<code>rnfdedekind(<i>nf</i>, T, pr)</code>
discriminant of relative extension	<code>rnfdisc(<i>nf</i>, T)</code>
pseudo-basis of $\mathbf{Z}_L$	<code>rnfpseudobasis(<i>nf</i>, T)</code>

General $\mathbf{Z}_K$ -modules: $M = [\text{matrix, vec. of ideals}] \subset L$

relative HNF / SNF	<code>nfhnf(<i>nf</i>, M), nfsnf</code>
multiple of $\det M$	<code>nfdetint(<i>nf</i>, M)</code>
HNF of M where $d = nfdetint(M)$	<code>nfhnfmod(<i>nf</i>, d)</code>
reduced basis for M	<code>rnfilllgram(<i>nf</i>, T, M)</code>
determinant of pseudo-matrix M	<code>rnfdet(<i>nf</i>, M)</code>
Steinitz class of M	<code>rnfsteinitz(<i>nf</i>, M)</code>
$\mathbf{Z}_K$ -basis of M if $\mathbf{Z}_K$ -free, or 0	<code>rnfhnfbasis(<i>bnf</i>, M)</code>
n -basis of M , or $(n + 1)$ -generating set	<code>rnfbasis(<i>bnf</i>, M)</code>
is M a free $\mathbf{Z}_K$ -module?	<code>rnfisfree(<i>bnf</i>, M)</code>

Based on an earlier version by Joseph H. Silverman

July 2026 v2.40. Copyright © 2026 K. Belabas

Permission is granted to make and distribute copies of this card provided the copyright and this permission notice are preserved on all copies.

Send comments and corrections to (Karim.Belabas@math.u-bordeaux.fr)

Associative Algebras

A is a general associative algebra given by a multiplication table mt (over $\mathbf{Q}$ or $\mathbf{F}_p$); represented by al from `algtableinit`.
create al from mt (over $\mathbf{F}_p$) `algtableinit(mt, {p = 0})`
group algebra $\mathbf{Q}[G]$ (or $\mathbf{F}_p[G]$) `alggroup(G, {p = 0})`
center of group algebra `alggrouppcenter(G, {p = 0})`
Properties
is (mt, p) OK for `algtableinit`? `algisassociative(mt, {p = 0})`
multiplication table mt `algmultable(al)`
dimension of A over prime subfield `algdim(al)`
characteristic of A `algchar(al)`
is A commutative? `algiscommutative(al)`
is A simple? `algissimple(al)`
is A semi-simple? `algissemisimple(al)`
center of A `algcenter(al)`
Jacobson radical of A `algradical(al)`
radical J and simple factors of A/J `algsimpledec(al)`

Operations on algebras

create A/I , I two-sided ideal `algquotient(al, I)`
create $A_1 \otimes A_2$ `algtensor(al1, al2)`
create subalgebra from basis B `algsubalg(al, B)`
quotients by ortho. central idempotents e `algcentralproj(al, e)`
isomorphic alg. with integral mult. table `algmakeintegral(mt)`
prime subalgebra of semi-simple A over $\mathbf{F}_p$ `algprimesubalg(al)`
find isomorphism $A \cong M_d(\mathbf{F}_q)$ `algsplit(al)`

Operations on lattices in algebras

lattice generated by cols. of M `alglathnf(al, M)`
... by the products xy , $x \in lat1$, $y \in lat2$ `alglatmul(al, lat1, lat2)`
sum $lat1 + lat2$ of the lattices `alglatadd(al, lat1, lat2)`
intersection $lat1 \cap lat2$ `alglatinter(al, lat1, lat2)`
test $lat1 \subset lat2$ `alglatsubset(al, lat1, lat2)`
generalized index $(lat2 : lat1)$ `alglatindex(al, lat1, lat2)`
 $\{x \in al \mid x \cdot lat1 \subset lat2\}$ `alglatlefttransporter(al, lat1, lat2)`
 $\{x \in al \mid lat1 \cdot x \subset lat2\}$ `alglatrighttransporter(al, lat1, lat2)`
test $x \in lat$ (set c = coord. of x) `alglatcontains(al, lat, x, {\&c})`
element of lat with coordinates c `alglatelement(al, lat, c)`

Operations on elements

$a + b$, $a - b$, $-a$ `algadd(al, a, b)`, `algsub`, `algneg`
 $a \times b$, a^2 `algmul(al, a, b)`, `algsqr`
 a^n , a^{-1} `algpow(al, a, n)`, `alginv`
is x invertible ? (then set $z = x^{-1}$) `alginv(al, x, {\&z})`
find z such that $x \times z = y$ `algdivl(al, x, y)`
find z such that $z \times x = y$ `algdivr(al, x, y)`
does z s.t. $x \times z = y$ exist? (set it) `algdivl(al, x, y, {\&z})`
matrix of $v \mapsto x \cdot v$ `algtomatrix(al, x)`
absolute norm `algnorm(al, x)`
absolute trace `algtrace(al, x)`
absolute char. polynomial `algcharpoly(al, x)`
given $a \in A$ and polynomial T , return $T(a)$ `algpoleval(al, T, a)`
random element in a box `algrandom(al, b)`

Central Simple Algebras

A is a central simple algebra over a number field K ; represented by al from `alginitt`; K is given by a nf structure.
create CSA from data `alginitt(B, C, {v}, {flag = 3})`
multiplication table over K $B = K$, $C = mt$
cyclic algebra $(L/K, \sigma, b)$ $B = rnf$, $C = [sigma, b]$
quaternion algebra $(a, b)_K$ $B = K$, $C = [a, b]$
matrix algebra $M_d(K)$ $B = K$, $C = d$
local Hasse invariants over K $B = K$, $C = [d, [PR, HF], HI]$
ramification set of quaternion algebra $B = K$, $C = [PR, HI]$
Hamilton quaternion algebra $(-1, -1)_{\mathbf{R}}$ $B = 1.$, $C = 1/2$
recover (a, b) if $al = (a, b)_K$ `algisquatalg(al)`
recompute al using current precision `algnewprec(al)`

Properties

type of al (mt , CSA) `algtype(al)`
dimension of A over $\mathbf{Q}$ `algdim(al, 1)`
dimension of al over its center K `algdim(al)`
degree of A ($= \sqrt{\dim_K A}$) `algdegree(al)`
 al a cyclic algebra $(L/K, \sigma, b)$; return σ `algaut(al)`
... return b `algb(al)`
... return L/K , as an rnf `algsplittingfield(al)`
split A over an extension of K `algsplittingdata(al)`
splitting field of A as an rnf over center `algsplittingfield(al)`
multiplication table over center `algrelmultable(al)`
places of K at which A ramifies `algramifiedplaces(al)`
Hasse invariants at finite places of K `alghassef(al)`
Hasse invariants at infinite places of K `alghassei(al)`
Hasse invariant at place v `alghasse(al, v)`
index of A over K (at place v) `algindex(al, {v})`
is al a division algebra? (at place v) `algisdivision(al, {v})`
is A ramified? (at place v) `algisramified(al, {v})`
is A split? (at place v) `algissplit(al, {v})`
is $A \cong A_2$? (at place v) `algisisom(al, al2, {v})`

Operations on elements

$x \in nf$ as an element of al `algeltfromnf(al, x)`
reduced norm `algnorm(al, x)`
reduced trace `algtrace(al, x)`
reduced char. polynomial `algcharpoly(al, x)`
express x on integral basis `algalgtobasis(al, x)`
convert x to algebraic form `algbasistoalg(al, x)`
express quaternion x on integral basis `algquattobasis(al, x)`
convert x to quaternion form `algbasistoquat(al, x)`
quaternion conjugate of x `alginvol(al, x)`
map $x \in A$ to $M_d(L)$, L split. field `algtomatrix(al, x)`
init mod pr map $\mathcal{O}_0 \rightarrow M_k(\mathbf{F}_q)$ `algmodprinit(al, pr, {v})`
project x to $M_k(\mathbf{F}_q)$ `algmodpr(al, x, modP)`
lift from $M_k(\mathbf{F}_q)$ to $\mathcal{O}_0$ `algmodprlift(al, x, modP)`
 $g \in A^\times$ s.t. $fa = gag^{-1}$ `algskolemnoether(al, a, fa)`

Orders

Z-basis of order $\mathcal{O}_0$ `algbasis(al)`
discriminant of order $\mathcal{O}_0$ `algdisc(al)`
Z-basis of natural order in terms $\mathcal{O}_0$'s basis `alginvbasis(al)`
Z-basis of Eichler order of level N `algeichlerbasis(al, N)`

Based on an earlier version by Joseph H. Silverman
July 2026 v2.40. Copyright © 2026 K. Belabas
Permission is granted to make and distribute copies of this card provided the
copyright and this permission notice are preserved on all copies.
Send comments and corrections to (Karim.Belabas@math.u-bordeaux.fr)